Real Option Pricing in a Fuzzy Stochastic Environment

I-ming Jiang*, Po-yuan Chen**, Chingliang Chang***

Abstract

This paper presents a fuzzy stochastic model for pricing the option to invest in an irreversible investment. The model considers both randomness and fuzziness in the value of the investment. Randomness states that the future value is always uncertain, and fuzziness states that the investor has his own subjective judgments on the system, which cannot be homogeneous and should be considered. Fuzziness is given by triangular fuzzy numbers, which are easily applied to the original stochastic model, McDonald and Siegel (1986). An optimality equation for the range of investment threshold is derived. It is shown that the optimal price range of the option to invest is the solution of Bellman equation by dynamic programming. The fuzzy goal is also presented for the permissible range of the investor's demand profits, which are discussed in numerical examples.

Keywords: Real Option, Fuzzy Stochastic Environment, Investor Subjective Judgment, Optimal Price Range

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1. Introduction

The option to invest in an irreversible investment has been valued by the continuous-time stochastic model of McDonald and Siegel (MS) (1986). Most investors make and sometimes revise their investment decisions continuously through time. Thus, the analysis of investment decisions should be devoted to the continuous-time problems. A stochastic process views the investment's future value as a variable that evolves over time, denoting that it is always uncertain. The irreversible investment is an investment that we cannot recover its initial expenditures all in case of the market conditions turn out to be worse than anticipated. The irreversible investment expenditure undermines the classical net present value rule due to the reason that when the investment is irreversible, the decision to invest can be postponed when the value of the investment does not exceed the investment threshold (Pindyck, 1991). The option to invest describes the option-nature of the investment. When investors make irreversible investment expenditures, they exercise their options to invest while give up the rights to wait for new information that might affect the timing of the investment. Viewed from the above perspectives, investment decisions share three important determinants such as the timing of the investment, irreversibility, and uncertainty in varying degrees. McDonald and Siegel (1986) contribute to recognize the important quantitative implications of interaction between the timing of the investment, irreversibility, and uncertainty.

In the MS model, the investment decisions are assumed to be made by homogeneous investors. This assumption implies that all investors have homogeneous judgments (homogeneity). Although Brennan (1979) and Stapleton and Subrahmanyam (1984) have pointed out that the assumption of investors with homogeneity is the basic economic paradigm, we still argue that there exist the heterogeneous judgments (heterogeneity) inherent with the investors to affect their investment decisions. De Bondt (1993), and Mankiw and Zeldes (1991) find empirical evidence of heterogeneity. One of the ways to describe heterogeneity is to regard the investors to be optimistic or pessimistic.

The optimism is similar to the idea of the overconfidence of Daniel, Hirshfeiler and Subrahmanyam (2001). The overconfident (optimistic) investors usually underestimate the uncertainty degree of the investment, leading to underestimate the investment threshold. In other words, the optimistic investors easily underestimate the trigger of the investment and make their decisions to invest. In the context of the fuzzy theory, the optimism is directly related to the lower membership degree (Peters, 2003). On the other hand, the pessimistic investors are especially likely to overestimate the threshold and defer the investment decisions. In terms of the theory of fuzzy sets, the pessimism is directly related to the higher membership degree

(Peters, 1996). Hence, the pessimistic and optimistic investors have heterogeneous references of the membership degree in the fuzzy set, and then their investment decisions are affected.

The above ideas of the optimism and pessimism allow us to explain the relation of the uncertainty to the investment. The pessimistic investors are likely to defer the investment in case that the uncertainty is highly unpredictable because they scare the failure of the investment. As a result of the pessimism, the uncertainty should be negatively related to the investment. Some authors, Caballero (1991), Leahy and Whited (1996), and Pindyck (1991), find that there exists the negative relationship between uncertainty and investment. In contrast to the pessimistic investors, the optimistic investors are likely to make the decision-makings to invest because they usually underestimate the high uncertainty, and look at the investment's success. As a result, there should be the positive relation of the uncertainty to the investment for the optimistic investors. Sarkar (2000, 2003) theoretically derives that the positive relationship between uncertainty and investment. The derivation can help explain why some investors invest more than others under uncertainty.

Basak (2000), Detemple and Murthy (1994), Harris and Raviv (1993), Mayshar (1983), Robinstein (1973), and Shefrin and Statman (1994) have treated heterogeneity for option pricing. One of the methods applied by the above authors is to treat heterogeneity by the fuzzy theory. Recently, Benninga and Mayshar (1997), and Robinstein (1994) begin to focus their attention on heterogeneity in pricing option. However, because the MS model is based on the basic economic paradigm, it has limited success in explaining the heterogeneity in the investment decision. Thus, there is a need to modify the MS model.

To the best of our knowledge, this paper first develops the fuzzy stochastic valuation model of the option to invest. In this model, we add the heterogeneity into the MS model. The heterogeneity is modeled by the membership degree of the triangular fuzzy function. We apply the technique of dynamic programming to derive closed-form solutions (Adda and Cooper, 2003; Dixit and Pindyck, 1994; Peskir and Shiryaev, 2006), and conduct a comparative-static analysis showing that the range of the values of the option to invest are sensitive to the investors' subjective judgments on the volatilities of both the value of the investment and the investment cost.

In the next section, we first construct a simple case, in which the value of the investment evolves according to the geometric Brownian motion, but the investment cost is known and fixed. We solve the optimal investment threshold by dynamic programming. In Section 3, we apply the fuzzy stochastic process to the simple case to define a closed interval of the value of the option to invest. And we extend the fuzzy stochastic process into the MS model. In Section 4, we approximate the fuzzy values of the option to invest in a numerical example and compute fuzzy goal. Finally, in Section 5, we give concluding remarks.

2. Simple Case

Consider an investor that is trying to invest in a firm or industry specific investment. The investment is completely irreversible — the investor cannot "uninvest" and recover the investment expenditure. The investment problem is to decide when it is optimal to undertake the investment. We assume that the investment cost, I, is known and fixed, but the value of the investment, V_t, follows a geometric Brownian motion of the form,

$$dV_t = \mu_V V_t dt + \sigma_V V_t dz_V \, , \tag{1}$$

where $\mu_V \geq 0$ is the instantaneous expected rates of the returns, $\sigma_V \geq 0$ is the instantaneous standard deviation, and dz_V is the increment of a Wiener process. Equation (1) implies that the current value of the investment is known, but the future values evolve according to the lognormal distribution with a variance that grows linearly with time horizon. Thus, although information arrives over time (the firm observes V changing), the future value of the investment is always uncertain.

We will be mostly concerned with the ways in which the investment decision is affected by the uncertainty. Thus, we focus attention on the case with uncertainty, that is, σ_V in equation (1) is greater than zero. The investment problem is to determine the point at which it is optimal to make an irreversible investment expenditure I in return for an investment whose value is V_t. Since V_t evolves stochastically, we will be unable to determine an investment timing. Thus, our investment rule will base on the optimal investment threshold V^* such that it is optimal to invest once $V_t \geq V^*$. Then, the investors receive the net payoff $V^* - I$ from investing. As we will see, a higher value of σ_V will result in a higher value V^*. When the optimal investment threshold is higher, fewer investments can exceed the threshold as an implication of waiting for the investment. Thus, higher uncertainty σ_V can create a greater value to waiting and thereby affect the investment timing. Following we introduce some Propositions, which enable us to solve the value of the option to invest using the dynamic programming.

Proposition 1: Let $F_S(V_t)$ denote the value of the option to invest at time t in the simple case. When $V_t \geq V^*$, stopping is optimal because the value of option to invest yields no cash flows up to the time T, the only return from holding the option to invest is its capital appreciation. Hence, in the case where $V_t < V^*$ (values of V_t for which it is not optimal to invest) the Bellman equation (Malliaris and Brock, 1982) is

$$\rho F dt = E_t(dF), \tag{2}$$

where E_t denotes the expectation at time t and ρ denotes a discount rate. Equation (2) indicates that over a time interval dt, the total expected return on the value of the option to invest, $\rho F dt$, is equal to its expected increment value, $E_t(dF)$.

Proposition 2: If we try the function $F_S(V_t) = AV_t^{\beta_1}$, then the optimal investment threshold V^*, β_1 and $F_S(V_t)$ at time t, can now be solved after imposing the appropriate boundary conditions including the initial, value-matching, and smooth-pasting conditions. The solutions can be written as follows:

$$V^* = \frac{\beta_1}{\beta_1 - 1} I, \tag{3}$$

$$\beta_1 = \frac{1}{2} - \frac{\mu_V}{\sigma_V^2} + \sqrt{\left(\frac{\mu_V}{\sigma_V^2} - \frac{1}{2}\right)^2 + 2\frac{\rho}{\sigma_V^2}} > 1, \tag{4}$$

$$A = \frac{1}{\beta_1}\left(\frac{\beta_1}{\beta_1 - 1} I\right)^{-(\beta_1 - 1)} = \frac{1}{\beta_1}\left(V^*\right)^{-(\beta_1 - 1)}, \tag{5}$$

and

$$F_S(V_t) = \begin{cases} V^* - I, & \text{for } V_t \geq V^*, \\ \left(\dfrac{1}{\beta_1 - 1} I\right)\left(\dfrac{V_t}{V^*}\right)^{\beta_1}, & \text{for } V_t < V^*. \end{cases} \tag{6}$$

Proof: See Appendix.

Equation (3) gives the solution of the optimal investment threshold V^*, and has the important point that since $\beta_1 > 1$, we have $\beta_1/(\beta_1 - 1) > 1$, and this shows that the optimal investment threshold is greater than the NPV's threshold, that is, $V^* > I$. Thus, uncertainty and irreversibility drive a wedge between the optimal investment threshold V^* and I. The size of the wedge is the factor $\beta_1/(\beta_1 - 1)$. Equation (4) shows two important implications. First, as σ_V increases, β_1 decreases, and therefore $\beta_1/(\beta_1-1)$ increases. The greater is the amount of the uncertainty over the future values of V, the larger is the wedge between V^* and I, that is, the larger is the excess return the investor will demand before it is willing to make the irreversible investment. Second, as r increases, β_1 decreases, so a higher r implies a larger wedge. Some results concerning β_1 are also informative. As $\sigma \to \infty$, we have $\beta_1 \to 1$ and $V^* \to \infty$, that is, the investor never invest if σ is infinite. Equation (5) gives the value of the constant A, which is sensitive to β_1 and V^*. Equation (6) indicates the binary optimal investment rule. If $V \geq V^*$, it is optimal to invest, and $F(V)$ is equal to the net payoff $V^* - I$. The most import point is that since $\beta_1 > 1$, we have $\beta_1/(\beta_1-1) > 1$ and $V^* > I$. Thus, the value of $F_S(V_t)$ is greater than zero. If $V < V^*$, the investor should wait rather than invest now, and growth in V at some future times will create a value to waiting, and increase the value of the option to invest. The solution enables us to further develop the fuzzy values of the option to invest in the next section.

3. Fuzzy Theory and Fuzzy Values of the Option to Invest

3.1 Fuzzy Theory

By adopting Yoshida (2003), we construct a model where the investment value process $\{V_t\}_{t=0}^{T}$ takes fuzzy values, and then represent the investors' subjective judgments using the membership degrees of the fuzzy values. For the sake of simplicity, the membership degrees are given by triangular fuzzy numbers. A fuzzy number is commonly defined by its corresponding membership function $\gamma:\mathbb{R}\rightarrow[0,\ 1]$ where $\mathbb{R}$ denotes all real numbers.

Assumption 1: Let $\{\alpha_t\}_{t=0}^{T}$ be an $\mathcal{M}_t$ -adapted stochastic process such that $0<\alpha(\omega)\le V_t(\omega)$ for all $\omega\in\Omega$. Then we represent a fuzzy stochastic process of the investment value $\{\tilde{V}_t\}_{t=0}^{T}$ by the following fuzzy random variables:

$$\tilde{V}_t(\omega)(x):=L\left(\left(x-V_t(\omega)\right)/\alpha_t(\omega)\right), \tag{7}$$

for $t\in\mathbb{T}$, $\omega\in\Omega$ and $x\in\mathbb{R}$, where $L(x):=\max\{1-|x|,\ \mathrm{x}>0\}$ $(x\in\mathbb{R})$ is the triangular-type shape function (Figure 1).

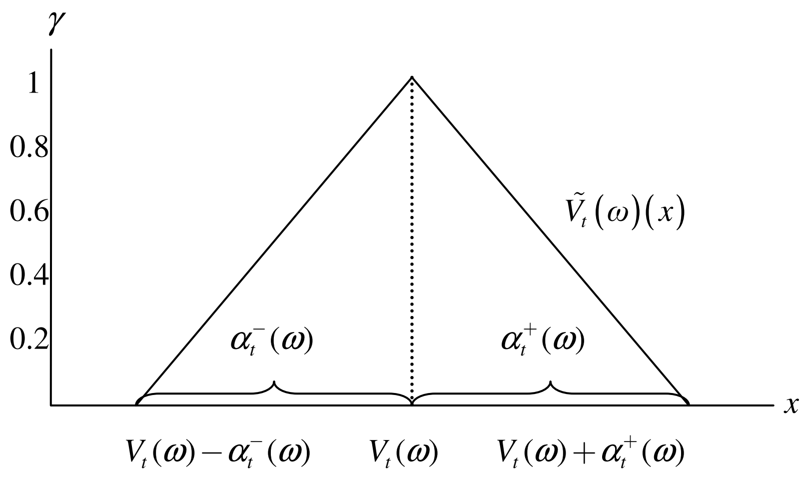

Figure 1. Fuzzy random variable $\tilde{V}_t(\omega)(x)$

Assumption 2: $\alpha_t(\omega)$ is a spread of triangular fuzzy numbers $\tilde{V}_t(\omega)$ and corresponds to the amount of fuzziness in the process. Then the γ-cuts of equation (7) are closed intervals

$$\tilde{V}_{t,\gamma}(\omega)=\left[V_t(\omega)-(1-\gamma)\alpha_t^-(\omega),\ V_t(\omega)+(1-\gamma)\alpha_t^+(\omega)\right],\ \ \omega\in\Omega\,, \tag{8}$$

and so we write the closed intervals by $\tilde{V}_{t,\gamma}(\omega):=V_t(\omega)\pm(1-\gamma)\alpha_t^{\pm}(\omega)$ for $\omega\in\Omega$, $t\in\mathbb{T}$ and $\gamma\in[0,\ 1]$.

Now we introduce the Assumption 3 from which we are able to simplify the formulas of Equation (8).

Assumption 3. The stochastic process $\alpha^{\pm}(\omega)$ is specified by $\alpha_t^{\pm}(\omega)=c^{\pm}V_t(\omega)$, $t=0,\ 1,\ 2,...,\ T,\ \omega\in\Omega$, where $c^{\pm}$ is constant satisfying $0<c^{\pm}<1$.

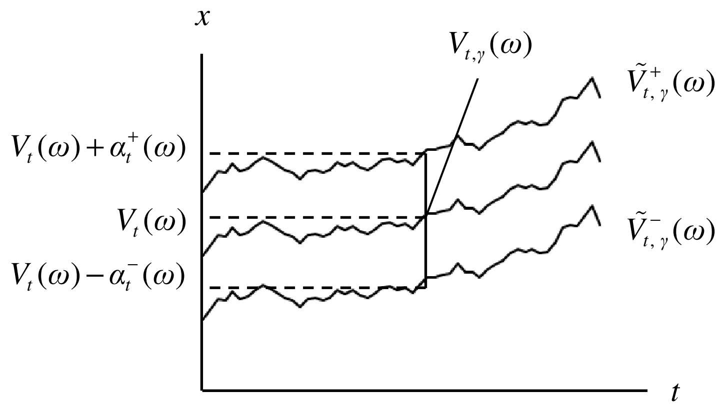

Figure 2. The stochastic process $\alpha^{\pm}(\omega)$

Assumption 3 is reasonable since $\alpha(\omega)$ measures a size of fuzziness and it should depend on the volatility σ and the investment value $V_t(\omega)$. But, in reality it is very difficult to estimate the volatility (Ross, 1999) (Figure 2). To overcome the difficulty, we use $c^{\pm}$ to represent the fuzziness of σ, which can be called as a fuzzy factor of the process. The fuzzy factor c represents the investor's subjective estimation of σ, while α represents the investor's subjective estimation of the value of the investment. Substituting $\alpha_t^{\pm}(\omega)=c^{\pm}V_t(\omega)$, where $0<c^+\leq c^-<1$, into equation (8) gives

$$\tilde{V}_{t,\gamma}^{\pm}(\omega)=\left(1\pm(1-\gamma)c^{\pm}\right)V_t(\omega)\,. \tag{9}$$

In brief, the fuzzy theory has great contribution in modeling the investors' heterogeneous judgments on the uncertainty because it can provide a range of heterogeneous fuzzy values to describe the set of subjective judgments without clear boundaries. The fuzzy logic performs well because the economic conditions change quickly and the investors always have their own judgments which are likely to be quite different from the others. In the next section, we introduce a fuzzified version of the values of the option to invest.

3.2 Fuzzy Values of the Option to Invest

Let us look at equation (9), because we assume the existence of heterogeneous judgments, the values of the investment cannot be just single. Hence, they should be a value range. Following we introduce Propositions to propose a closed-form solution for the fuzzy values of the option to invest.

Proposition 3: Let the fuzzy stochastic process of the Bellman equation be $\rho\tilde{F}^{\pm}dt=E(d\tilde{F}^{\pm})$, and the fuzzy values of the option to invest is $\tilde{F}_{S}^{\pm}\left(V_t\right)=A^{\pm}\left(V_t\right)^{\beta_1}$, which must satisfy the initial, value-matching, and smooth-pasting boundary conditions. Then, we can find the two unknowns—the constant $A^{\pm}$, and the fuzzy optimal investment threshold $\tilde{V}^{*\pm}$.

$$\tilde{V}^{*\pm} = \frac{\beta_1}{\beta_1 - \left(1 \pm (1-\gamma) c^{\pm}\right)} I . \tag{10}$$

and

$$A^{\pm} = \frac{\left(1 \pm (1-\gamma) c^{\pm}\right)}{\beta_1} \left(\tilde{V}^{*\pm}\right)^{-(\beta_1 - 1)} . \tag{11}$$

We will determine $\tilde{F}_S^{\pm}(V_t)$ in much the same way that we do in Proposition 2. We concentrate on the case where continuation is optimal for $V_t < \tilde{V}^{*\pm}$, and stopping is optimal for $V_t \geq \tilde{V}^{*\pm}$. Then, we find the following solution for $\tilde{F}_S^{\pm}(V_t)$:

$$\tilde{F}_S^{\pm}(V_t) = \begin{cases} \tilde{V}^{*\pm} - I, & \text{if} \quad V_t \geq \tilde{V}^{*\pm}, \\ \left(\dfrac{\left(1 \pm (1-\gamma) c^{\pm}\right)}{\beta_1 - \left(1 \pm (1-\gamma) c^{\pm}\right)} I \right) \left(\dfrac{V_t}{\tilde{V}^{*\pm}} \right)^{\beta_1}, & \text{if} \quad V_t < \tilde{V}^{*\pm}. \end{cases} \tag{12}$$

Proof: See Appendix.

Let us compare equation (3) with equation (10), we assume that the future values of the investment are random, and there exists the heterogeneous judgments on the uncertainty. Hence, the value is suitable to be fuzzified. This makes the reasonable results for the formula of the optimal investment threshold should be a range of the values not just a single value anymore. To propose the upper and lower bounds to describe the heterogeneity is the one of the major contributions in our paper. Using the value bound to describe the threshold is due to the reason that the values of the investment are always uncertain. We assume that the optima investment rule depends on whether the values of the investment V_t exceed the optimal threshold V^*. Hence, the threshold also has the uncertain nature about its actual value. Equation (10) shows that V^* under the stochastic simple case is replaced with $\tilde{V}^{*\pm}$ under the fuzzy stochastic model. The "$\pm$" for the equations comes from the fuzzy random variables gives the closed-form of the optimal investment threshold, and has the important point that since $\beta_1 > 1$, $c^{\pm} \in (0,\ 1)$, and $\gamma \in [0,\ 1]$, we have $\beta_1 / \left(\beta_1 - \left(1 \pm (1-\gamma) c^{\pm}\right)\right) > 1$ and $\tilde{V}^{*\pm} > I$. Equation (11) indicates that the constant A is sensitive to β_1, $\tilde{V}^{*\pm}$, γ, and $c^{\pm}$.

Equation (12) gives the fuzzy values of the option to invest and the optimal investment rule, that is, the fuzzy optimal investment threshold $\tilde{V}^{*\pm}$. There are two important implications of $\tilde{V}^{*\pm}$. First, the pessimistic investors do the investment only when the value of the investment is much larger than the upper bound of the threshold, that is, $V_t > \tilde{V}^{*+}$. Second, the optimistic investors are likely to require the lower bound of the threshold for investment. $\tilde{F}_S^{\pm}(V_t)$ for both investors is equal to the net payoff $\tilde{V}^{*\pm} - I$. The most import point is that since $\beta_1 > 1$, $0 \le \gamma \le 1$ and $0 < c^{\pm} < 1$, we have $\beta_1 / \left(\beta_1 - \left(1 \pm (1-\lambda) c^{\pm}\right)\right) > 1$ and $\tilde{V}^{*\pm} > I$. Thus, the fuzzy values of $\tilde{F}_S^{\pm}(V_t)$ is greater than zero. If $V_t < \tilde{V}^{*-}$, for both the pessimistic and optimistic investors, waiting is better than investing. However, if $\tilde{V}^{*-} < V_t < \tilde{V}^{*+}$, the investment decisions of the investors are vague. The optimistic investors will exercise their option to invest while the pessimistic investors will not. Then, we will extend the simple case where the investment cost I is known and fixed to the MS model where the investment cost I also follows the geometric Brownian motion.

3.3 Extension to the MS Model

The MS model values the option to invest assuming that the investment cost I is also random and follows the geometric Brownian motion:

$$dI_t = \mu_I I_t dt + \sigma_I I_t dz_I \,, \tag{13}$$

where μ_I denote the instantaneous expected cost rates of the investment, the volatility σ_I represents the instantaneous standard deviation of the rates of the investment cost, dz_I denotes the increment of a Wiener process.

The optimal investment is to invest when $C_t\left(=V_t/I_t\right)$ reaches a barrier $\left\{C_t^*\right\}$, the expected present value of the payoff at time t is[1]

$$E_t\left\{I_{t'}\left[C^*-1\right]e^{-\mu t'}\right\}=\left[C^*-1\right]E_t\left\{I_{t'}e^{-\mu t'}\right\}, \tag{14}$$

where t' is the date at which C_t first reaches the boundary C^*, μ is the risk-adjusted discount rate, and the expectation is taken over the joint density of I_t and the first-passage times for C_t. Here, the evaluation of equation (14) is relegated to the Appendix of McDonald and Siegel (1986). From the Appendix, the value of the option to invest at time t is

[1] In the special case where the investment opportunity is infinitely lived, it is possible to solve for the maximized value of equation (14) explicitly. When $T=\infty$, it is possible to remove calendar time from the problem. Hence, C^* cannot depend on t, so $C_t^*=C^*$ for all t.

$$F_{\mathrm{MS}}\left(C_t\right)=\begin{cases}\left(C_t-1\right)I_t, & \text{if} \quad C_t \geq C^* \\ \left(C^*-1\right)I_t\left(\dfrac{C_t}{C^*}\right)^{\varepsilon}, & \text{if} \quad C_t < C^*\end{cases}, \tag{15}$$

and

$$C^*=\frac{\varepsilon}{\left(\varepsilon-1\right)}, \tag{16}$$

$$\varepsilon=\sqrt{\left(\frac{\mu_V-\mu_I}{\sigma^2}-\frac{1}{2}\right)^2+\frac{2\left(\mu-\mu_I\right)}{\sigma^2}}+\left(\frac{1}{2}-\frac{\mu_V-\mu_I}{\sigma^2}\right), \tag{17}$$

where $\sigma^2=\sigma_V^2+\sigma_I^2-2\rho_{VI}\sigma_V\sigma_I$, and ρ_{VI} is the instantaneous correlation between the rates of V_t and I_t.

With the standard asset pricing model, the risk premium earned by the investment is proportional to the riskiness of the investment, that is, $\hat{\mu}_i-r=\varphi\rho_{im}\sigma_i$ where r is the risk-free rate, φ is the market price of risk, and ρ_{im} is the correlation between the rate of the return on the investment and that on the market portfolio. Let μ be the required expected rate of return and hence the equilibrium expected rate of on the option to invest, will be given by

$$\mu=\varepsilon\hat{\mu}_V+\left(1-\varepsilon\right)\hat{\mu}_I, \tag{18}$$

where $\hat{\mu}_V$ and $\hat{\mu}_I$ are determined by $\hat{\mu}_i-r=\varphi\rho_{im}\sigma_i$. By equating the required expected rate of return with the actual expected rate of return computed in the article of McDonald and Siegel (1986) yields a quadratic equation in ε:

$$\mu=\varepsilon\mu_V+\left(1-\varepsilon\right)\mu_I+\frac{1}{2}\varepsilon\left(\varepsilon-1\right)\sigma^2. \tag{19}$$

and ε is the root of the above quadratic equation

$$\varepsilon = \sqrt{\left(\frac{\delta_I - \delta_V}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\delta_I}{\sigma^2}} + \left(\frac{1}{2} - \frac{\delta_I - \delta_V}{\sigma^2}\right), \tag{20}$$

where $\delta_V = \hat{\mu}_V - \mu_V$ and $\delta_I = \hat{\mu}_I - \mu_I$. $\delta_V > 0$ insures that $\varepsilon > 1$. For the value of the investment V to be bounded, we must have $\hat{\mu}_V > \mu_V$, and denote the difference $\hat{\mu}_V - \mu_V$ by δ_V. When some underlying parameter changes, the equilibrium relationship $\delta_V = \hat{\mu}_V - \mu_V$ must continue to hold. Now, when the σ_V of V increases, $\hat{\mu}_V$ must increase. If δ_V is a fundamental market constant, then σ_V must change one for one with $\hat{\mu}_V$. However, if σ_V is a fundamental market constant, then δ_V must adjust. When we study the effects of changes in σ_V on the investor's investment decision, the answer will depend on which of the above viewpoints we adopt. Generally, we will take δ_V to be the basic parameter and let σ_V adjust. Following we introduce Proposition 4, which enables us to extend the MS model into the fuzzy stochastic model.

Proposition 4: In the MS model, if $C_t \left(= V_t / I_t\right) \geq C^*$, the investment undertaken, and deferred otherwise. In other words, the investment rule can also take the form of a optimal threshold $V_t \geq I_t C^*$ such that it is optimal to invest once $V_t \geq V_t^* \left(= I_t C^*\right)$. Thus, the optimal investment threshold should be divided by I. Let $\tilde{V}_\tau^{*\pm} = \tilde{V}^{*\pm} / I$ where $\tilde{V}^{*\pm}$ is shown in equation (10) and we apply $\beta_1 = \varepsilon$ to give the fuzzy value of the option to invest for MS model

$$\tilde{F}_{\mathrm{MS}}^{\pm}(C_t)=\begin{cases}\left(C^{*\pm}-1\right)I_t, & \text{if} \quad C_t \geq C^{*\pm}, \\ \left(\dfrac{\left(1\pm(1-\gamma)c^{\pm}\right)}{\varepsilon-\left(1\pm(1-\gamma)c^{\pm}\right)}\right)I_t\left(\dfrac{C_t}{C^{*\pm}}\right)^{\beta_1}, & \text{if} \quad C_t < C^{*\pm}.\end{cases} \tag{21}$$

$$C^{*\pm}=\frac{\varepsilon}{\varepsilon-\left(1\pm(1-\gamma)c^{\pm}\right)}. \tag{22}$$

Proof: See Appendix.

Equation (21) is the solution for the value of the option to invest by fuzzifing the MS model. While the stochastic process models the randomness of the future values of the investment, the investors have their own subjective judgments on the investment. So, the subjectivities cannot be ignored. This makes the needs to incorporate them into the existing stochastic process by fuzziness. Then, we implement numerical example and compute the fuzzy goal.

4. Numerical Example and Fuzzy Goals

In this section, the fuzzy stochastic model is implemented to compute the closed-form for the values of the option to invest of the MS model. We also conduct the comparative-state analysis to see the sensitivity of the value of the option to invest. Furthermore, we show how the valuation method works if the fuzzy expectations are considered taking the investor's subjective utility function into account.

4.1 Numerical Example

We implement the fuzzy stochastic model to approximate the spread of fuzzy value of the option to invest. As a base case, we take an asymmetric triangular-type fuzzy function that accounts for the investors' subjective judgments on the volatilities of the values of investment. The high volatility makes the investors shift to the left due to the reason that they would decide to wait for new information (Figure 3).

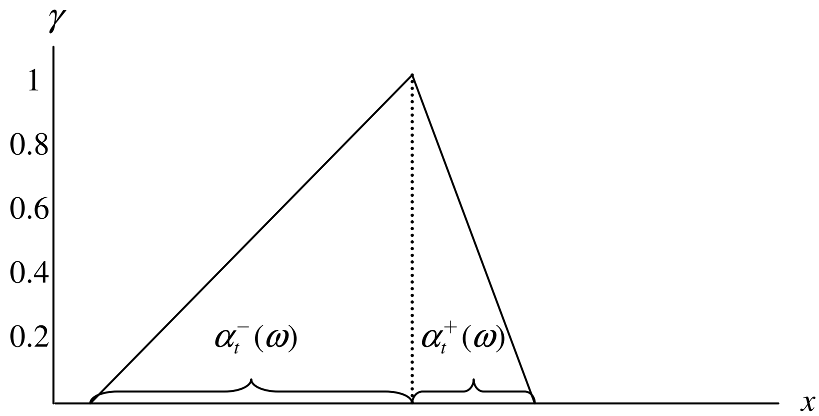

Figure 3. Asymmetric triangular fuzzy numbers $\alpha_t^{\pm}(\omega)$

In order to capture the phenomenon, c^- is fixed at 50% and c^+ at 10% throughout the approximation. In practice, more realistic values for $c^{\pm}$ are based on the individual investor's subjective estimation of the volatility. We take 0.90 for γ-cut that accounts for the investor's confidence over the optimal expected value of $\tilde{V}$ under randomness and fuzziness.

It is noted that the formula of β_1 would not change in the fuzzy stochastic model. To determine the value of β_1, we refer to ε of McDonald and Siegel (1986) with $\sigma_V^2 = \sigma_I^2 = 0.04$, $\delta_V = \delta_I = 0.10$ and ρ_{VI} = -0.5, 0.0 and 0.5. If the investment value V_0 is a present value, a reasonable parameter for σ_V is the average standard deviation for unlevered equity in the United States, which is about 0.20. δ_V measures the extent to which the expected value increase in V_0 alone fail to compensate investors for the risk of value changes in V_0, which is set at 0.10. Appropriate choices for I_0 are less clear. If the investment cost is nonstochastic, δ_I should be the risk-free rate. If I_0 is systematically risky, but $\mu_I = 0$, then it would be greater.

Table 1. A comparison of the value of the option to invest under the fuzzy stochastic model and the MS model ($V_0 = I_0 = 1$)

	Fuzzy stochastic model			MS model		
δ_V		0.10				
ρ_{VI}	-0.5	0.0	0.5	-0.5	0.0	0.5
σ_V^2, σ_I^2						
0.01	[0.13, 0.15]	[0.11, 0.13]	[0.08, 0.09]	0.14	0.12	0.08
0.04	[0.24, 0.31]	[0.20, 0.25]	[0.15, 0.18]	0.27	0.23	0.16
0.30	[0.42, 0.84]	[0.40, 0.68]	[0.33, 0.49]	0.60	0.52	0.40
δ_I						
0.01	[0.16, 0.20]	[0.12, 0.14]	[0.07, 0.08]	0.18	0.13	0.07
0.10	[0.24, 0.31]	[0.20, 0.25]	[0.15, 0.18]	0.27	0.23	0.16
0.25	[0.34, 0.51]	[0.33, 0.47]	[0.31, 0.43]	0.42	0.39	0.36

Note: Base case parameters are $\sigma_V^2 = \sigma_I^2 = 0.04$ and $\delta_V = \delta_I = 0.10$. For fuzzy stochastic model, entries are computed using (9), (20) and (22) in the current paper. For the MS model, entries are computed using (4) and (12) in the text.

Table 1 compares the value of the option to invest under the fuzzy stochastic model with the MS model. The entries in Table 1 represents the loss per dollar of V if the investment were undertaken at $V_0/I_0 = 1$, rather than waiting until the optimal time. The value of the option to invest can never exceed V_0, so 1.00 is an upper bound for the entries in Table 1. For example, if $\sigma_V^2 = \sigma_I^2 = 0.01$, $\rho_{VI} = 0$, and $\delta_V = \delta_I = 0.10$, then the value of the option to invest is worth of 12 percent of V_0. To conduct a comparative-static analysis, we let the other parameters stay the same but the volatility increasing to $\sigma_V^2 = \sigma_I^2 = 0.04$, then the value of the option would be 23 percent of V_0. Same to the above two examples, as the confidence level $\gamma = 0.90$, $c^- = 50\%$ and $c^+ = 10\%$, the fuzzy value interval is [0.20, 0.25], noting that 0.23 is included in the range. If the value of the option to invest is below the lower bound 0.20, then investing is likely to be postponed. If the value is above the upper bound 0.25, then investors will undertake the investment. If the value is below 0.25, the pessimistic investors will decide to defer the investment due to the reason that they need the larger value of the investment to overcome the investment failure. However, if the value lies between 0.20 and 0.25, then the investment decisions of the investors are vague. Because the optimistic investors are likely to decide to invest, the pessimistic investors will not. For narrowing the closed interval, the investor can increase the confidence level to decrease the width of the interval. Additionally, we approximate a wide spread of parameters. It demonstrates that the value of the MS model stay at any position of the closed interval.

Figure 4 conducts the sensitivity analysis of the fuzzy values of the option to invest for these parameters, $\beta_1 = 2.16$, $\delta_V = \delta_I = 0.10$, $\rho_{VI} = 0.0$, $c^- = 0.5$, $c^+ = 0.1$, $\gamma = 0.9$, and also for . $\sigma_V^2 = \sigma_I^2 = 0.04$ and $\sigma_V^2 = \sigma_I^2 = 0.25$. In each case, the tangency point of $\tilde{F}_{\mathrm{MS}}^{\pm}(V_t)$ with the line $C_t\left(= V_t / I_t\right) = 1$ gives the values of the fuzzy optimal investment threshold $C^{*\pm} = \varepsilon \big/ \left(\varepsilon - \left(1 \pm (1-\gamma) c^{\pm} \right) \right)$ or $V_{\text{MS}}^{*\pm} = \varepsilon I_t \big/ \left(\varepsilon - \left(1 \pm (1-\gamma) c^{\pm} \right) \right)$. The value also shows that the simple NPV must be modified to include the opportunity cost of investing now rather than waiting. The opportunity cost is exactly $\tilde{F}_{\text{MS}}^{\pm}(V_t)$. When $V_t < V_{\text{MS}}^{*\pm}$, $\tilde{F}_{MS}^{\pm}(V_t) > V_{\text{MS}}^{*\pm} - I_t$ and therefore $V_{\text{MS}}^{*\pm} < I_t + \tilde{F}_{MS}^{\pm}(V_t)$: the value of the investment is less than its full cost, the direct cost I_t plus the opportunity cost $\tilde{F}_{\text{MS}}^{\pm}(V_t)$. (when $V_{\text{MS}}^{*\pm} = I_t$ and $\tilde{F}_{\text{MS}}^{\pm}(V_t) = 0$ for $V_{\text{MS}}^{*\pm} \leq V_t$.) Note that $\tilde{F}_{\text{MS}}^{\pm}(V_t)$ increase when σ increases, as does the fuzzy optimal investment threshold $V_{\text{MS}}^{*\pm}$. As a result, growth in V_t creates a value to waiting, and increases the value of the option to invest.

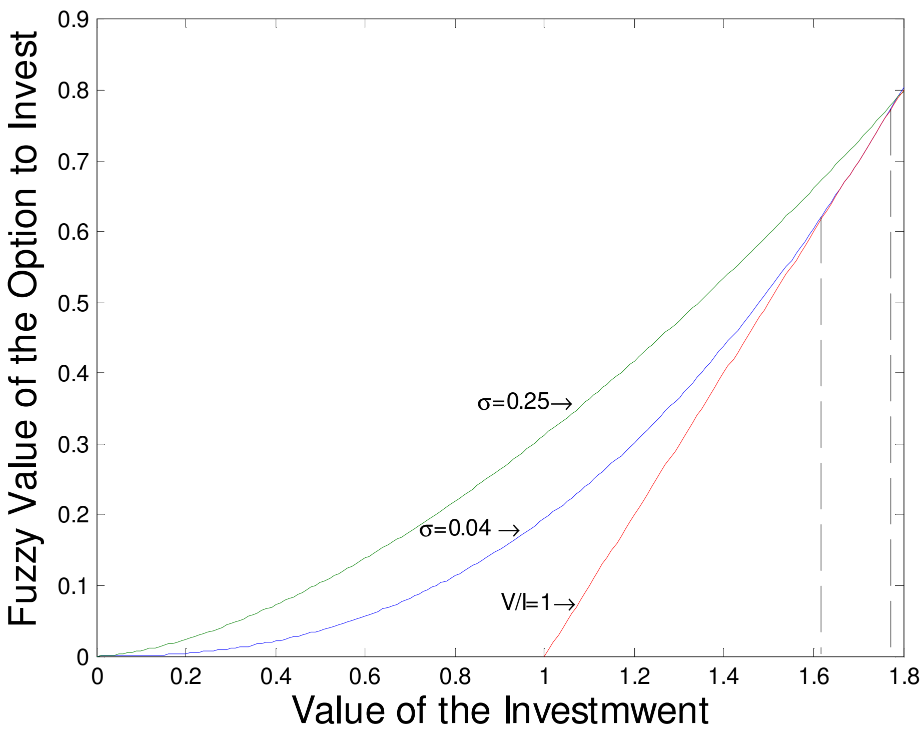

Figure 4. Fuzzy value of the option to invest, $\tilde{F}_{\text{MS}}^{\pm}(V_t)$

Table 2 shows that the investment in the above first example would be optimal if

V_0/I_0 reaches 1.37, and in the second example at 1.86. Table 2 clearly shows that the level of V_0/I_0 at which the investment is optimal is typically much greater than 1.00. Comparing with $V_0/I_0 = 1.37$, the fuzzy stochastic model shows that the lower and upper bounds are 1.44 and 1.48. Comparing with $V_0/I_0 = 1.37$, they are 1,79 and 1.88.

Table 2. Value of benefit relative to investment cost (V_0/I_0) at which investment is optimal

	Fuzzy stochastic model			MS model		
δ_V			0.10			
ρ_{VI}	-0.5	0.0	0.5	-0.5	0.0	0.5
σ_V^2, σ_I^2						
0.01	[1.44, 1.48]	[1.35, 1.38]	[1.23, 1.25]	1.47	1.37	1.25
0.04	[2.02, 2.16]	[1.79, 1.88]	[1.52, 1.57]	2.13	1.86	1.56
0.30	[5.01, 6.70]	[4.03, 4.98]	[2.87, 3.26]	6.34	4.79	3.19
δ_I						
0.01	[1.59, 1.65]	[1.40, 1.44]	[1.20, 1.22]	1.64	1.43	1.22
0.10	[2.02, 2.16]	[1.79, 1.88]	[1.52, 1.57]	2.13	1.86	1.56
0.25	[3.00, 3.44]	[2.80, 3.16]	[2.58, 2.86]	3.35	3.09	2.81

Note: Base case parameters are $\sigma_V^2 = \sigma_I^2 = 0.04$ and $\delta_V = \delta_I = 0.10$. For fuzzy stochastic model, entries are computed using (9), (20) and (22) in the current paper. For the MS model, entries are computed using (4) and (12) in the text.

Further compare the value of the option to invest with the value under uncertainty. Table 3 presents the percentage of the value of the option to invest which is due to uncertainty. As the table shows, increase in δ_V, holding δ_I constant, increases the uncertainty. As the table shows, increase in percentage of the value due to uncertainty. When $\delta_V \geq \delta_I$, all of the value is due to uncertainty, since otherwise waiting would be suboptimal. The percentages presented by the MS model stay at the closed intervals of the current proposed model.

Table 3. Percentage of value of the option to invest which is due to uncertainty

	Fuzzy stochastic model	MS model
σ^2	0.04	
$\delta_V = 0.02$	[8.7, 9.6]	9.4
$\delta_V = 0.04$	[19.8, 20.3]	20.1
$\delta_V = 0.06$	[36.3, 36.5]	36.4
$\delta_V = 0.08$	[61.8, 61.9]	61.9
$\delta_V = 0.10$	100.0	100.0

Note: All of these calculations assume that $V_0 = I_0 = 1$ and $\delta_I = 0.10$. For fuzzy stochastic model, entries are computed using (9), (20) and (22) in the current paper. For the MS model, entries are computed using (4), (12) and (13) in the text.

4.2 Fuzzy Goal

The fuzzy goal means a kind of utility for expected values of the option to invest, and it represents the investor's subjective judgment from the idea of Bellam and Zadeh (1970). We follow Agliardi and Agliardi (2009) and show how the valuation method works if fuzzy expectations are considered taking the investor's subjective utility function into account. Let φ denote a fuzzy goal, that is a continuous and increasing function from $[0, +\infty)$ to $[0, 1]$, such that $\varphi(0)=0$ and $\varphi(x)=1$ as $x \to \infty$. The fuzzy expectation under φ of a fuzzy number X is defined by

$$E(X) = \sup_{x\in R}\{X(x),\ \varphi(x)\}, \tag{22}$$

and a rational expected value is defined as a real number where the fuzzy expectation attains the supremum. The following argument is confined to determining the expected fuzzy values of the option to invest.

Let the investor's fuzzy goal, φ, be given. Define a grade γ^* by

$$\gamma^{V^+} := \sup\left\{\gamma \in [0,\ 1] \mid \varphi_\gamma^- \le \tilde{F}_{\gamma,\,\mathrm{MS}}^{V^+}(V_0,\ 0)\right\}, \tag{23}$$

where $\varphi_\gamma = \left[\varphi_\gamma^-,\ \infty\right)$ for $\gamma \in (0,\ 1)$ and the supremum of the empty set is understood to be 0. From equation (23) and the continuity of φ and $\tilde{F}_{\gamma,\,\mathrm{MS}}^{V+}$, we can easily check

$$\varphi_{\gamma^{V+}}^- = \tilde{F}_{\gamma^{V+},\,\mathrm{MS}}^{V+}(V_0,\ 0). \tag{24}$$

Proposition 5: Under the fuzzy expectation generated by possibility measures, equation (22), the following (5a), and the following (5a) hold.

(5a) The membership degree of the fuzzy expectation of the values of the option to invest $\tilde{F}_{\mathrm{MS}}^{V}$ is given by

$$\gamma^{V^+} = \tilde{E}\left(\tilde{F}_{\mathrm{MS}}^{V}(V_0,\ 0)\right). \tag{25}$$

(5b) Further, the rational expected values of the option to invest is given by

$$x^{V^+} = \varphi_{\gamma^{V^+}}^- . \tag{26}$$

Since the fuzzy expectation (22) is defined by possibility measures, equation (26) gives an upper bound on rational expected values of the option to invest. Therefore, similar to equation (26), another membership degree which gives a lower bound on optimal expected values of the option to invest can be defined as follows:

$$x^{V^-} = \varphi_{\gamma^{V^-}}^- , \tag{27}$$

where γ^{V^-} is defined by

$$\gamma^{V^-} := \sup\left\{\gamma \in [0,\ 1] \mid \varphi_\gamma^- \le \tilde{F}_{\gamma,\,\mathrm{MS}}^{V^-}(V_0,\ 0)\right\}, \tag{28}$$

and it satisfies

$$\varphi_{\gamma^{V-}}^{-} = \tilde{F}_{\gamma^{V-},\,\mathrm{MS}}^{V-}\left(V_0,\ 0\right). \tag{29}$$

Hence, from equations (23), (24), (25), (26), (27), (28) and (29), it can be easily checked that the interval $\left[x^{V-},\ x^{V+}\right]$ is written as:

$$\left[x_*,\ x^*\right] = \left\{x \in \mathbb{R} \mid \varphi\left(x\right) \le \tilde{F}_{\gamma,\,\mathrm{MS}}^{V}\left(V_0,\ 0\right)\left(x\right)\right\}, \tag{30}$$

which is the range of values x such that the membership degree of the expected fuzzy values of the option to invest, $\tilde{F}_{\gamma,\,\mathrm{MS}}^{V}\left(V_0,\ 0\right)\left(x\right)$, is greater than the degree of the investor's satisfaction, $\varphi\left(x\right)$. Therefore, $\left[x^{V-},\ x^{V+}\right]$ means the investor's permissible range of expected values under his fuzzy goal φ.

Example: Let φ denote a fuzzy goal, that is continuous and increasing function from $\left[0,\ +\infty\right)$, such that $\varphi\left(0\right)=0$ and $\varphi\left(x\right)\to 1$ as $x \to +\infty$ .Assume that the fuzzy goal, $\varphi_c^i\left(x\right)$ ($i=1,\ 2$), are defined as follows:

$$\varphi^i\left(x\right) = \begin{cases} 1-e^{-\lambda_i x}, & x \ge 0, \\ 0, & x < 0. \end{cases} \tag{31}$$

where λ_i can be a view of the investor's risk preference, or we say that the investors have a constant absolute risk aversion (CARA) utility with the parameter λ_i. Equation (31) indicates that the fuzzy goal is $\varphi^i\left(x\right)=1-e^{-\lambda_i x}$ if $x \ge 0$ and 0 otherwise. In nature, when the investors become more risk aversion, they tend to increase the reference of the membership degree in the fuzzy set, thus causing the narrow interval of the fuzzy values.

Going back to the base case we discuss in Section 4. The parameters of this example is $\delta_V = \delta_I = 0.10$, $\rho_{VI} = 0.0$, $\sigma_V^2 = \sigma_I^2 = 0.04$, $\sigma^2 = 0.08$, $\varepsilon = 2.16$, $c^- = 0.50$, $c^+ = 0.10$, $\lambda_1 = 0.9$, and $\lambda_2 = 0.4$. Higher value of λ represents more degree of risk aversion.

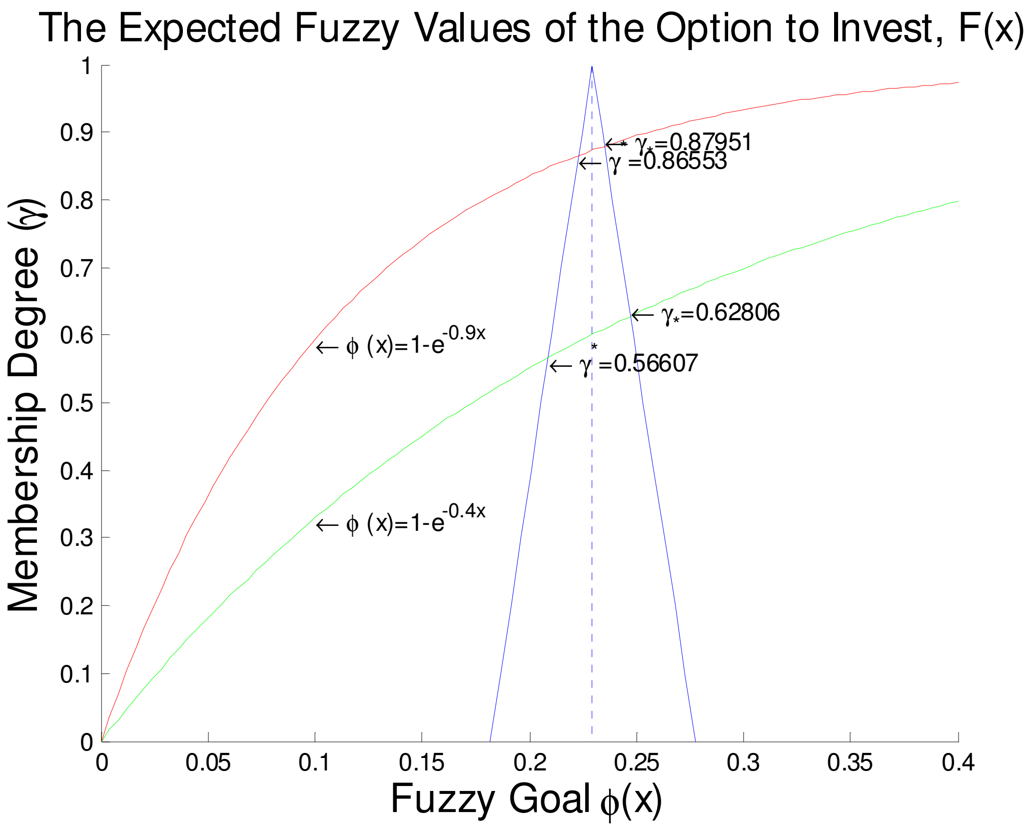

Figure 5. Membership degree (γ), expected fuzzy values of the option to invest optimal fuzzy value $\tilde{F}_{\text{MS}}^{\pm}(V_t)$, and fuzzy goal $\varphi(x)$

From Figure 5, we find that the membership level of the fuzzy expectation of fuzzy values are $[\gamma_*, \gamma^*] = [0.8655,\ 0.8795]$, and $[\gamma_*, \gamma^*] = [0.5661,\ 0.6281]$ for fuzzy goal equals to equation (27) with $\lambda_1 = 0.9$ and $\lambda_2 = 0.4$, respectively, this membership level means that the degrees of the investor's satisfaction in valuing. The corresponding permissible range for the fuzzy values of the option to invest under the fuzzy goals are $\left[x_{*},\ x^{*}\right] = [0.2229,\ 0.2351]$, and $\left[x_{*},\ x^{*}\right] = [0.2087,\ 0.2473]$, respectively. It is clear that the interval difference in the higher risk aversion ($\lambda_1 = 0.9$) is 0.0122, which is smaller than 0.0386 in the lower risk aversion ($\lambda_2 = 0.4$). This evidence indicates that an increase in λ in the CARA utility will narrow the range of the expected fuzzy values. Another useful interpretation is that the narrow range will easily cause the investors with high risk aversion to defer their investment, that is, they believe that waiting is better than investing, as we discuss in the Introduction and Equation (12). As fuzzy factor c^{-} and c^{-} approach zero, the fuzzy values would converge to the MS model.

5. Concluding Remarks

This paper introduces fuzziness to the stochastic process for evaluating the option to invest in an irreversible investment. The existing stochastic process simply deals with the randomness as uncertainty over the volatility of the value of the investment, in which the value of the investment is assumed to follow the geometric Brownian motion. In effect, the investors' heterogeneities on the volatility are likely to be quite different, but they cannot be modeled by the existing stochastic process. Our first addition of the investors' subjective judgments modeled by the fuzziness to the stochastic model is one of our major contributions. Thus, our model can consider the fuzziness as uncertainty to define the fuzzy values of the investment. The fuzziness is subject to the changes of the membership degrees such that the investors can increase the membership degree to decrease the interval of the fuzzy values. This fact points to the influences of the optimism and pessimism in making the investment decisions.

We also contribute showing how to incorporate triangular fuzzy numbers into the existing stochastic process. The incorporation is a subjective new development in evaluating the option to invest. The development can help explain the properties of the investment decisions of the optimistic and pessimistic investors. We also hope that the methods for the numerical computation of the optimal grade of the investor's confidence, and the corresponding optimal interval of the fuzzy values will be employed in practice. If necessary, the further research may develop a defuzzified method to find a crisp value from the interval of the fuzzy values.

Appendix:

1. Proof of Proposition 2

We expand dF of the Bellman equation $\rho F dt = E_t(dF)$ using Ito's Lemma, and we use primes to denote derivatives, for example, $F_V = dF/dV$ and $F_{VV} = d^2F/dV^2$. Then

$$dF = F_V dV + \frac{1}{2} F_{VV} (dV)^2 . \tag{A1}$$

The above expression is substituted with equation (1) for dV , and $E(dz)$ is zero. This gives

$$E(dF) = \mu_V V F_V dt + \frac{1}{2} \sigma_V^2 V^2 F_{VV} dt . \tag{A2}$$

After dividing through dt , the Bellman equation becomes:

$$\frac{1}{2} \sigma_V^2 V_t^2 F_{VV} + \mu_V V_t F_V - \rho F_{\mathrm{S}}(V_t) = 0 . \tag{A3}$$

Additionally, the value of the option to invest $F_{\mathrm{S}}(V_t)$ must satisfy the following boundary conditions:[2]

$$F_{\mathrm{S}}(0) = 0 , \tag{A4}$$

$$F_{\mathrm{S}}(V^*) = V^* - I , \tag{A5}$$

$$F_{V,\mathrm{S}}(V^*) = 1 . \tag{A6}$$

Since the second-order homogeneous differential equation (A3) is linear in the dependent variable F and its derivatives, its general solution can be expressed as a

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[2] Condition (A4) is the initial condition which indicates that if V_t is zero, then the value of the option to invest will be zero. Condition (A5) is the value-matching condition which indicates that if the firm invests, then the firm receives a net payoff $V^* - I$. Condition (A6) is the smooth-pasting condition which indicates that if $F_{\mathrm{S}}(V_t)$ were not continuous and smooth at the investment threshold V^* , then one could do better by exercising as a different point.

linear combination of any two independent solutions. If we try the function AV_t^{β}, we see by substitution that it satisfies the equation provides β is a root of the quadratic equation. When $F_S(V_t) = AV_t^{\beta}$, we can obtain $F_V = \beta AV_t^{\beta-1}$ and $F_{VV} = \beta(\beta-1)AV_t^{\beta-2}$. Substituting them into equation (A3) gives

$$\frac{1}{2}\sigma_V^2\beta(\beta-1)+\mu_V\beta-r=0. \tag{A7}$$

The two roots are

$$\beta_1=\frac{1}{2}-\frac{\mu_V}{\sigma_V^2}+\sqrt{\left(\frac{\mu_V}{\sigma_V^2}-\frac{1}{2}\right)^2+2\frac{\rho}{\sigma_V^2}}\,(>1), \tag{A8}$$

and

$$\beta_2=\frac{1}{2}-\frac{\mu_V}{\sigma_V^2}-\sqrt{\left(\frac{\mu_V}{\sigma_V^2}-\frac{1}{2}\right)^2+2\frac{r}{\sigma_V^2}}\,(<0). \tag{A9}$$

The value of β cannot be less than 1 to yield the positive optimal investment threshold. Note that equation (A3) is a second-order differential equation, but there are three boundary conditions that must be satisfied. The reason is that although the position of the first boundary ($V=0$) is known, the position of the second boundary is not, in other words, the "free boundary" V^* must be determined as part of the solution. That needs the third condition.

To satisfy the boundary condition (A4), the solution for the value of the option to invest must take the form

$$F_S(V_t) = AV_t^{\beta_1}\ [3],\qquad \text{(A10)}$$

where A is a constant that will be determined, and $\beta_1 > 1$ is a known constant whose value depends on the parameters μ, σ, and ρ of the differential equation. The remaining boundary conditions, (A5) and (A6), can be used to solve the two remaining unknowns — constant A, and the optimal investment threshold V^* at which it is optimal to invest. By substituting (A10) into (A5) and (A6) and rearranging, we obtain that

$$V^* = \frac{\beta_1}{\beta_1 - 1} I, \qquad \text{(A11)}$$

and

$$A = \frac{1}{\beta_1}\left(\frac{\beta_1}{\beta_1 - 1} I\right)^{-(\beta_1 - 1)} = \frac{1}{\beta_1}\left(V^*\right)^{-(\beta_1 - 1)}. \qquad \text{(A12)}$$

By plugging equation (A11) and (A12) into the $F_S(V_t) = AV_t^{\beta_1}$ and rearranging, we find the value of the option to invest:

$$F_S(V_t) = \begin{cases} V^* - I, & \text{if } V_t \ge V^*, \\ \left(\frac{1}{\beta_1 - 1} I\right)\left(\frac{V_t}{V^*}\right)^{\beta_1}, & \text{if } V_t < V^*. \end{cases} \qquad \text{(A13)}$$

2. Proof of Proposition 3

Equation (9) shows $\tilde{V}^{\pm}_{t,\gamma}(\omega) = \left(1 \pm (1-\gamma)c^{\pm}\right)V_t(\omega)$. The Ito's Lemma is used as the usual rules of calculus to define this differential in terms of first-order changes in

[3] The general solution should be $F_S(V_t) = A_1 V_t^{\beta_1} + A_2 V_t^{\beta_2}$. When $\beta_2 < 0$ and $V = 0$, the value of the option to invest will become ∞. This violates the boundary condition (A4), $F_S(0) = 0$. So, A_2 must be zero leaving the solution $F_S(V_t) = AV_t^{\beta_1}$.

$\tilde{V}^{\pm}$:

$$d\tilde{V}_t^{\pm} = \tilde{V}_t^{\pm}\left(\mu_V dt + \sigma_V dz_V\right). \tag{A14}$$

Now we are ready to examine the investor's optimal investment rule. For algebraic simplicity we will again assume an exogenously specified discount rate, and use the dynamic programming approach. Let the fuzzy stochastic process of the Bellman equation be $\rho\tilde{F}^{\pm}dt = E_t(d\tilde{F}^{\pm})$. By expanding $d\tilde{F}^{\pm}$ with Ito's Lemma, we obtain $E_t\left(d\tilde{F}^{\pm}\right) = \mu\tilde{V}^{\pm}\tilde{F}_{\tilde{V}}^{\pm}dt + \frac{1}{2}\sigma^2\tilde{V}^{\pm^2}\tilde{F}_{\tilde{V}\tilde{V}}^{\pm}dt$. So, the Bellman equation can be rewritten as $\frac{1}{2}\sigma_V^2V^2\tilde{F}_{\tilde{V}\tilde{V}}^{\pm} + \mu_V VF\tilde{F}_{\tilde{V}}^{\pm} - r\tilde{F}^{\pm} = 0$. In addition, the fuzzy values of the option to invest $\tilde{F}_{\mathrm{S}}^{\pm}\left(V_t\right)$ must satisfy the following boundary conditions: the initial condition $\tilde{F}_{\mathrm{S}}^{\pm}\left(0\right) = 0$, the value-matching condition $\tilde{F}_{\mathrm{S}}^{\pm}\left(\tilde{V}^{*\pm}\right) = \tilde{V}^{*\pm} - I$, and the smooth-pasting condition $\tilde{F}_{V,\,\mathrm{S}}^{\pm}\left(\tilde{V}^{*\pm}\right) = \left(1 \pm \left(1-\gamma\right)c^{\pm}\right)$. This condition means that if $\tilde{F}_V^{\pm}$ are not continuous and smooth at the fuzzy optimal investment thresholds $\tilde{V}^{*\pm}$, the investors could invest better at a different point (Dixit, 1988).

By adopting the similar way to (A10), we find two unknowns — constant $A^{\pm}$ and the optimal investment threshold $\tilde{V}^{\pm *}$:

$$\tilde{V}^{*\pm} = \frac{\beta_1}{\beta_1 - \left(1 \pm \left(1-\gamma\right)c^{\pm}\right)} I, \tag{A15}$$

and

$$A^{\pm} = \frac{\left(1 \pm \left(1-\gamma\right)c^{\pm}\right)}{\beta_1}\left(\tilde{V}^{*\pm}\right)^{-(\beta_1 - 1)}. \tag{A16}$$

We then plug equation (A15) and (A16) into the $\tilde{F}_S^{\pm}(V_t) = A^{\pm}(V_t)^{\beta_1}$. Thus,

$$\tilde{F}_S^{\pm}(V_t) = \begin{cases} \tilde{V}^{*\pm} - I, & \text{if} \quad V_t \geq \tilde{V}^{*\pm}, \\ \left(\dfrac{\left(1 \pm (1-\gamma)c^{\pm}\right)}{\beta_1 - \left(1 \pm (1-\gamma)c^{\pm}\right)} I \right) \left(\dfrac{V_t}{\tilde{V}^{*\pm}} \right)^{\beta_1}, & \text{if} \quad V_t < \tilde{V}^{*\pm}. \end{cases} \tag{A17}$$

3. Proof of Proposition 4

Under equation (14), we first write down the heterogeneous fuzzy values of the option to invest:

$$\tilde{F}_{MS}^{\pm}(C_t) = \left(C^{*\pm} - 1\right) I_t \left(\frac{C_t}{C^{*\pm}} \right)^{\varepsilon}, \tag{A18}$$

and using the similar method for equation (A15) to find the optimal investment threshold $C^{*\pm}$:

$$C^{*\pm} = \frac{\beta_1}{\beta_1 - \left(1 \pm (1-\gamma)c^{\pm}\right)}. \tag{A19}$$

Because of the adjustment discount rate in market equilibrium, now equation (A19) can be rewritten by applying $\beta_1 = \varepsilon$ to obtain the formula $C^{*\pm} = \dfrac{\varepsilon}{\varepsilon - \left(1 \pm (1-\gamma)c^{\pm}\right)}$, and simply plugging this formula to find the solution for the fuzzy values of the option to invest at time t in equation (A17):

$$\tilde{F}_{MS}^{\pm}(C_t) = \begin{cases} \left(C^{*\pm} - 1\right) I_t, & \text{if} \quad C_t \geq C^{*\pm}, \\ \left(\dfrac{\left(1 \pm (1-\gamma)c^{\pm}\right)}{\varepsilon - \left(1 \pm (1-\gamma)c^{\pm}\right)} \right) I_t \left(\dfrac{C_t}{C^{*\pm}} \right)^{\beta_1}, & \text{if} \quad C_t < C^{*\pm}. \end{cases} \tag{A20}$$

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