Economic Design of a Rectifying Inspection Sampling Procedure for Single Specification Limit Products

E-Ping Huang and Chiuh-Cheng Chyu*

Department of Industrial Engineering and Management
Yuan-Ze University
135 Yuan-tung Road, Chung-Li, 320, Taiwan, R.O.C.
Corresponding author*: iehshsu@saturn.yzu.edu.tw

Abstract: This paper develops a rectifying inspection sampling plan for single specification limit products under a Bayesian framework with the objective of minimizing the expected total cost due to imperfect items introduced to the manufacturer's production and sales systems. The model involves a two-stage decision: (1) determining the optimal sample size, and (2) after obtaining sampling information, taking an action that is between no more inspection and 100 percent inspection. The model assumes that the quality characteristic of the key component is exponentially distributed with an unknown mean. Using gamma distribution as the conjugate prior, a closed form decision criterion for the second stage can be derived, and the computational complexity will be greatly reduced. The attribute sampling model under the same probability distribution and cost structure is also derived and analyzed. Applications and numerical results using an LED electronic component for both sampling models are presented.

Keywords: Bayesian decision analysis, single specification limit, variable sampling plan, attribute sampling plan, conjugate prior distributions, rectifying inspection.

1. INTRODUCTION

In this paper we propose a Bayesian decision model that is useful in the quality control issue: whether an incoming lot should be subject to no inspection, 100 percent inspection, or acceptance sampling inspection. The total material cost to a producer includes both the price and the quality cost incurred by imperfect material in the producer's production and sales system In particular, we consider a case of rectifying inspection sampling, in which any non-conforming units in the lot will be replaced with a conforming one, including the units that cause malfunction products. Rectifying inspection sampling model is frequently used when manufacturing costs are high (Anderson et al. 2001). Common applications of rectifying inspection sampling are in semiconductor manufacturing and electronic products manufacturing, where maintaining control of the process is difficult and individual items have relatively high cost. In such cases, suppliers are often required to provide an estimated number of nonconforming units in outgoing lots. These estimates can be obtained from data that were collected during acceptance sampling. Greenberg and Stockes (1992) provides an efficient predictor for nonconforming units using the information obtained in rectification.

It is well known that the items produced by the same production process vary in quality due to some inevitable random factors in production, such as variation in materials and human and machine operations (Tang, 1992). The distribution of the quality characteristic of items is defined as the process distribution, which characterizes the performance of the produced items. Bayesian inference provides a formal mechanism for a producer to assess the unknown process distribution of the purchased items using his experience with a supplier's items and the sampling results obtained by in-plant inspections. Furthermore, Bayesian approach is capable of integrating the cost-quality relations with corresponding probabilities, which will be updated by the current information. In designing a Bayesian inspection sampling procedure, it is more convenient to use a conjugate prior distribution, since the computational complexity will be greatly reduced.

Deming (1982) discussed an (*n*, *c*) rectifying attributes sampling plan relative to two different cost setups, (k_1, k_2), where *n* is the sample size, *c* is the acceptance number, k_1 is the cost per unit to inspect an item, and k_2 is the cost per unit of a nonconforming item that is either placed in an assembly that fails or subsequently fails after entering the stream of commerce. Usually, the k_2 cost is much higher than the k_1 inspection cost. Moskowitz and Tang (1992) used the cost structure proposed by Schmidt et al. (1974) to develop a Bayesian variables acceptance sampling model with the following probability assumptions: the performance variable has a normal distribution with an unknown mean, which is assumed normally distributed as well. Tagaras (1994) studied a similar cost structure under the same probability assumptions, but assumed that the inspection was destructive, and thus the cost of inspection per unit would be greater than the cost of rejection per unit. Yeh and Van (1997) developed a Bayesian double-variable sampling model with the polynomial loss function under the same probability assumptions.

The purpose of this paper is to develop a rectifying inspection sampling procedure based on an economical consideration that can be applied to the case of single specification limit. Our literature review indicates that many studies were on double specification limit, but very few were on single specification limit. The models developed in this paper use the same cost structure as Deming's model, and contain two procedures: a variable sampling procedure and its corresponding attribute sampling procedure.

The remainder of this paper is organized as follows. In section 2, the variable and attribute sampling models are formulated based on the Bayesian decision rule. Section 3 presents an application example and numerical results. Section 5 concludes the paper.

2. BAYESIAN DECISION MODEL

This paper develops rectifying inspection sampling procedures for single specification limit products using a Bayesian approach with the objective to minimize the expected total cost. The procedures are similar to Deming's model (1982), but differ in the decision-making approach and the use of sampling information. Deming's model has been discussed in the models of 0-1 attribute (Lorezen, 1985; Papadakis, 1985; Barlow and Zhang, 1986; Chyu and Wu, 2002) and double specification limits (Chyu and Yu, 2006). This model uses rectifying inspection; rectifying inspection refers to a procedure whereby a lot rejected by sampling inspection is 100% inspected. It is also assumed that this inspection is 100% effective, and all nonconforming items discovered during inspection will be replaced by conforming units. In addition, if a nonconforming item is placed on the assembly line and results in a bad product, the product can be disassembled and the nonconforming item will be replaced with a conforming item. We furthermore assume that a spare lot comes along with the purchased lot, and the items in both lots will have the same quality. If a nonconforming item is found, then extra inspections on the items in the spare lot must be done until a conforming item is found. Such inspection cost will be charged to the manufacturer.

The model involves a two-stage decision. In the first stage, the objective is to determine the optimal sample size. After observing the sampling outcome, the second-stage objective is to decide whether to stop inspection and send the remaining items of the lot into assembly, or to continue to inspect the remainder of the lot. We refer to the decision "stop inspection" as "a_1," and the decision "inspect all" as "a_2."

The following are notations used in this paper:

N	: total number of components in the lot
n	: sample size
$\underline{x}$	: sampling data, $\underline{x} = (x_1, ..., x_n)$
y	: number of nonconforming items in the sample
k_1	: inspection cost per item
k_2	: product failure cost per item
a_1	: "stop inspection" decision at the second stage
a_2	: "inspect all" decision at the second stage
$R(y)$	: number of extra inspections to obtain y conforming items
Y_{N-n}	: number of nonconforming items in the remainer of the lot
$R(Y_{N-n})$	: number of extra inspections to obtain Y_{N-n} conforming items

Assume that at the second stage of the model, the sampling information are ($n, \underline{x}, y$). The loss due to decisions "a_1" and "a_2" are as follows, respectively.

$$L(a_1, n, \underline{x}, y) = n \cdot k_1 + R(y) \cdot k_1 + Y_{N-n} \cdot k_2 + R(Y_{N-n}) \cdot k_1 \quad (1)$$

$$L(a_2, n, \underline{x}, y) = n \cdot k_1 + R(y) \cdot k_1 + (N-n) \cdot k_1 + r(Y_{N-n}) \cdot k_1 \quad (2)$$

If we take the expected loss as the criterion to select action, then we conclude the following:

if $E(Y_{N-n} \mid n, \underline{x}, y) \cdot k_2 \le (N-n) \cdot k_1$, action a_1 is not inferior to action a_2. Otherwise, action a_2 is taken. According to the Bayesian decision theory, the optimal sample size must satisfy the following equation:

$$\underset{0\le n\le N}{Min}\, C(n) = \underset{0\le n\le N}{Min} \int_{\underline{x}} \sum_{y} \left[Min\left(L(a_1, n, \underline{x}, y), L(a_2, n, \underline{x}, y) \right) \right] \cdot \Pr\{y \mid n, \underline{x}\} \cdot f(\underline{x}) d\underline{x} \tag{3}$$

2.1 Variable Sampling Model

The model developed in this section is an extension of Deming's model for single specification limit case. For simplicity, we will focus the study on the lower bound case; i.e., (u,∞). The other case $(0,u)$ can be solved by a similar technique.

The components in the purchase lot are assumed to be manufactured under the same statistical quality control process and have the following distributions,

$X_1, X_2, ..., X_N \mid W = w \sim i.i.d.\ b + \exp(w)$, where $b > 0$ is a guaranteed performance value.

Under this assumption, the probability of a component being conforming given $W = w$ is

$\mathrm{P}(w) = \Pr\{X_k \ge u - b \mid W = w\} = e^{-u'w}$, where $u' = u - b$.

The unconditional probability is $P(W) = e^{-u'W}$, which is a random variable. If the prior distribution of W is Gamma (α, β), the probability of a component being conforming is

$$E[P(W)] = E[e^{-u'W}] = \left(\frac{\beta}{\beta + u'} \right)^{\alpha} \tag{4}$$

The cumulative distribution function (cdf) of $P(W)$ is

$$F_w(p) = \Pr\{P(W) \le p\} = \Pr\{e^{-u'W} \le p\} = \Pr\{W \ge \frac{\ln p}{-u'}\} = 1 - \int_0^{\frac{\ln p}{-u'}} \frac{\beta^{\alpha}}{\Gamma(\alpha)} w^{\alpha-1} e^{-\beta w} dw \tag{5}$$

In addition, the probability density function (pdf) of $P(W)$ is

$$f_w(p) = F_w{}'(p) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \cdot \left(\frac{\ln p}{-u'} \right)^{\alpha-1} \cdot e^{-\beta \cdot \left(\frac{\ln p}{-u'} \right)} \cdot \frac{1}{p \cdot u'},\ 0 < p < 1 \tag{6}$$

Applying the Bayes' theorem, we can obtain that if $W \sim$ Gamma(α, β), then the posterior of W depends on the simpling data, ($x_1, ..., x_n$), and the total number of nonconforming units, y, only through the sample size and the sum of performance measurements of the samples. That is,

$h(w \mid x_1, ..., x_n, y) = h(w \mid x_1, ..., x_n) = h(w \mid n, s)$, where $s = x_1 + ... + x_n - n \cdot b$ and $W \mid n, s \sim Gamma(\alpha + n, \beta + s)$.

By algebraic operations, the expected total cost corresponding to sample size n (equation (3)) can be simplified as follows.

$$C(n) = n \cdot k_1 + N \cdot E\left(\frac{1}{P(W)} - 1 \right) \cdot k_1 + (N - n) \cdot \int_0^{\infty} Min\left[\left(1 - E(P(W) \mid n, s)\right) \cdot k_2, k_1 \right] \cdot g(s \mid n) \cdot ds, \tag{7}$$

where $E\left(\frac{1}{P(W)} \right) = \left(\frac{\beta}{\beta - u'} \right)^{\alpha}$ and $E(P(W) \mid n, s) = \left(\frac{\beta + s}{\beta + s + u'} \right)^{\alpha + n}$. Clearly, action a_1 is preferred if $1 - E(P(W) \mid n, s) \le k_1 / k_2$; otherwise, action a_2 will be preferred. It is clear that $1 - E(P(W) \mid n, s)$ decreases as the value of s increases. After applying some algebraic operations, we can conclude the following:

If $s > s^* = \frac{u' \cdot \Delta}{1 - \Delta} - \beta$, a_1 is taken; otherwise a_2 will be taken, where $\Delta = \left(1 - k_1 / k_2\right)^{\frac{1}{\alpha + n}}$.

2.2 Computational Issue

It is difficult to accurately compute the integral part of equation (7) because the sum of the performance variables in the samples, $S = \sum_{k=1}^{n} (X_k - b)$, has a pdf over the range $(0,\infty)$, as shown in equation (8).

$$g\left(s \middle| n\right) = \frac{\Gamma(\alpha + n)}{\Gamma(\alpha) \cdot \Gamma(n)} \cdot \frac{s^{n-1} \cdot \beta^{\alpha}}{(\beta + s)^{n + \alpha}},\ s > 0 \tag{8}$$

By letting $Q = \beta \cdot (S+\beta)$, the transformed random variable Q has a distribution $Beta(\alpha, n)$ as shown below.

$$h(q \mid n) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)\cdot\Gamma(n)} \cdot q^{\alpha-1} \cdot (1-q)^{n-1},\ 0<q<1 \tag{9}$$

The integral part can be transformed into the following equation, and accurately computed by the Simpson method with 3/8 rule.

$$\int_0^1 Min[(1-\left(\frac{\beta}{\beta+u'q}\right)^{\alpha+n}) \cdot k_2, k_1] \cdot h(q \mid n) \cdot dq = k_1 \cdot \int_{q^*}^1 h(q \mid n) \cdot dq \quad + k_2 \cdot \left[\int_0^{q^*} (1-\left(\frac{\beta}{\beta+u' \cdot q}\right)^{\alpha+n}) \cdot h(q \mid n) dq\right], \tag{10}$$

where $q^* = \dfrac{\beta}{\beta+s^*} \ = \dfrac{1-\Delta}{u' \cdot \Delta} \cdot \beta$. According to our computational results, the function C(n) behaves like U-shape as a whole, but its values fluctuate slightly up and down in the local sense. Therefore, a good statregy to find an optimal sample size and the minimum expected total cost is the following:

> Start with k = 0. Compute C ($n = 10 \cdot k$) for k = 1, 2, … Stop when the value of C($10 \cdot k$) increases three times consecutively or $N \leq\ 10 \cdot k$. Find the lowest value of C(($10 \cdot k$). The sample size corresponds to the minimum cost in the range $[10 \cdot (k-1)\ ), 10 \cdot (k+1)\ )]$ is the solution wanted. If k = 0, the search range becomes [0, 10]; on the other hand, if k = N/10, then the search range becomes [N-10, N].

2.3 Attribute Sampling Model

In this section, a 0-1 attribute sampling model based on the same probability distribution and cost structure is derived. In general, the attribute sampling model carries less information but is simpler to use than the variable sampling model.

Let $Y_k = 1$ if the kth component is nonconforming and $Y_k = 0$ otherwise. Clearly, the probability of a component being conforming is Pr{ $Y_k = 0$ } = Pr{ $X_k > u'$ } = $E(P(W))$ = $\left(\dfrac{\beta}{\beta+u'}\right)^{\alpha}$.

The cdf and the pdf of P(W) were given in equations (5) and (6), respectively. Let $Y = \sum_{k=1}^{n} Y_k$ and y is the realization number of Y. Clearly, the conditional distribution of Y, given W = w, is Binomial (n, P(w) = $e^{-u'w}$). For simplicity, we denote P(W) as P. The following properties of $E(P \mid n, y)$ are useful in analyzing the attribute sampling model and reducing computations.

Property 1: $E(P \mid n, y)$ increases in n with y fixed.

Proof: $E(P \mid n+1, y) = \int_0^1 p \cdot f(p \mid n+1, y) dp = = \dfrac{E(P^2 \mid n, y)}{E(P \mid n, y)} \geq E(P \mid n, y)$, since $E(P^2 \mid n, y) \geq (E(P \mid n, y))^2$.

Thus, $E(P \mid n, y)$ increases in the sample size n with y fixed.

Property 2: $E(P \mid n, y)$ decreases in y with n fixed.

Proof: $E(P \mid n, y+1) = \dfrac{E(1-P \mid n, y)}{E((1-P)/P \mid n.y)} \leq E(P \mid n, y)$. Because $(1-P)/P$ and P are negatively correlated, $E(1-P \mid n, y) =\ E(P \cdot (1-P)/P \mid n, y) \leq E(P \mid n, y) \cdot E((1-P)/P \mid n, y)$.

Similarly, by algebraic operations, the objective function value of the attribute model corresponding to sample size n is simplified as follows.

$$C(n) = n \cdot k_1 + N \cdot E((\frac{1}{P})-1) \cdot k_1 + (N-n) \cdot \sum_{y=0}^{n} \left[Min\left((1-E(P \mid n, y) \cdot k_2, k_1\right)\right] \cdot \Pr\{y \mid n\} \tag{11}$$

By property 2, if $1-E(P \mid n, 0) \geq k_1/k_2$, then action a_2 is taken regardless of the value of y. If $1-E(P \mid n, n) \leq k_1/k_2$, then action a_1 is taken regardless of the value of y. However, if $1-E(P \mid n, 0) \leq k_1/k_2$ and $1-E(P \mid n, n) > k_1/k_2$, there exists a critical value c*, 0 ≤ c* ≤ n, such that action a_1 is taken when 0 ≤ y≤ c*, and action a_2 is taken when y > c*. In addition, according to property 1, as sample size grows, c* does not decrease. Thus, properties 1 and 2 are helpful in facilitating the computations in the last part of equation (11). Property 3 assists in checking the accuracy of the computations.

After some algebraic operations, we obtain $E(P \mid n, y) = \left(\frac{n-y+1}{n+2}\right) \cdot \frac{E(f_w(Q) \mid Beta(n-y+2, y+1))}{E(f_w(Q) \mid Beta(n-y+1, y+1))}$ and $\Pr\{y \mid n\} = (1/n+1) \cdot E(f_w(Q) \mid Beta(n-y+1, y+1))$. Here, $E(f_w(Q) \mid Beta(a,b)) = \int_0^1 f_w(q) \cdot \frac{\Gamma(a+b)}{\Gamma(a) \cdot \Gamma(b)} \cdot q^{a-1} \cdot (1-q)^{b-1} dq$, where $f_w(q)$ has the form of equation (6) and Q is *beta* (a, b). Such transformations will facilitate the computations when the Simpson's 3/8 rule is applied to the last part of equation (11). Finally, the algorithm to find the optimal sample size in section 2.2 can also be applied to this attribute sampling model.

3. APPLICATIONS AND NUMERICAL RESULTS

A certain electronic product contains an electronic device, which functions as a light signal. The light strength of this device must meet the specification (2.0, ∞) K-lumin. Relevant data of this electronic device manufactured by supplier A is as follows:

The production process centers the quality characteristic measurement at 2.1883 K-lumin. The center of the process is subject to fluctuations due to machine or human operations, and the standard deviation is estimated to be 0.06281 K-lumin. Collected information shows that the performance measurements of the devices manufacured by supplier A have never fallen below b = 1.98 K-lumin. Moreover, the statistical distribution plot indicates that choosing the form, $\left(\frac{\beta}{\beta+x-b}\right)^{\alpha}$, as the tail probability for the performance measurement of a device (i.e., Pr{ $X_j > x$ }), is acceptable. Thus, we take the probability model in section 2.1 for the quality characteristic of the items. Under this assumption, $E\left(\frac{1}{W}\right) = \frac{\beta}{\alpha-1}$ = 2.1883 – 1.98 = 0.2083, $\left(Var(\frac{1}{W})\right)^{1/2} = \frac{\beta^2}{(\alpha-1)^2 \cdot (\alpha-2)}$ = 0.06281. Using the numerical method, we obtain $\alpha \approx 13$ and $\beta \approx 2.5$. The probability of a device being conforming is $E(P(W))$ = 0.9016. Furthermore, we suppose N = 600, k_1 = 10.2 , and k_2 = 91.5. The ratio would then be 1- k_1/k_2 = 0.89 < 0.9016. Using the variable sampling model (see Table 3.1), the optimal sample size n^* = 39, and the critical value $b + \bar{s}^*$ = 2.1413 < 2.1883. If the average performance measurement of the samples $b + \bar{s} \leq 2.1413$, then no more inspection should be made; otherwise, the remainder of the lot should be inspected. Moreover, if the attribute sampling model is used, the acceptance criteria are (n^*, c^*) = (44, 7). The sample size of the attribute sampling model is usually larger than that of the variable sampling model because the latter carries more sampling information than the former. The expected cost per item under the variable sampling model is $9.847, which is less than that under the attribute sampling model, $10.10.

Suppose another supplier B has a production equipment which can manufacture items with guarranteed quality characteristic value b = 1.975 K-lumin. In order to provide the same conforming probability as supplier A, supplier B centers his production process measurement at $b + E\left(\frac{1}{W}\right) = 2.253$. Such a machine parameter setting may incur more set-up cost. The variation (standard deviation) of the process center is estimated to be $\left(Var(\frac{1}{W})\right)^{1/2}$ = 0.0982. Furthermore, stastistical analysis on the sampling data indicates that the probability model assumption of section 2.1 is acceptable as well. Table 3.2 shows that the acceptance criteria are $(n^*, b+\bar{s}^*)$ = (33, 2.174), with the expected cost per item equal to $9.478, which is less than supplier A. If the purchase prices of both suppliers are the same, then supplier B should be chosen based on an economical point of view.

Table 3.1 Sampling procedures for process centered at 2.188 with variation 0.0628 and b = 1.98

	Variable sampling model				Attribute sampling model		
N	Expected cost/unit	n^*	$\bar{s}^*$	$b+\bar{s}^*$	Expected cost/unit	n^*	c*

600	9.847	39	0.161	2.141	10.100	44	7
700	9.832	43	0.162	2.142	10.084	53	8
800	9.819	47	0.163	2.143	10.069	71	10

Table 3.2 Sampling procedures for process centered at 2.253 with variation 0.0982 and b = 1.975

	Variable sampling model				Attribute sampling model		
N	Expected cost/unit	n*	$\bar{s}$*	$b+\bar{s}$*	Expected cost/unit	n*	c*
600	9.478	33	0.199	2.174	9.732	35	6
700	9.462	36	0.200	2.175	9.717	52	8
800	9.449	40	0.202	2.177	9.705	70	10

4. CONCLUSION

In this paper, a rectifying inspection sampling model for single specification limit based on Bayesian decision thoery is developed. The objective of this model is to minimize the expected total cost due to imperfect items introduced into the manufacturer’s production and sales systems. Moreover, this model is useful in selecting the best supplier. There has been a great deal of research on acceptance sampling procedures but very few consider the use of Bayesian approach in the case of single specification limit. In this study, the quality characteristic of the key component is assumed to be exponentially distributed with an unknown mean. Using gamma distribution as the conjugate prior, the computational complexity of this model will be greatly reduced, and the decision criteria can be obtained shortly. Such makes the model useful in practical application. Meanwhile, the attribute sampling model under the same probability distribution and cost structure is derived and compared with the original variable sampling model. Numerical results suggests that the variable sampling model be used based on an economical view point, even though the attribute sampling model is simpler and easier used in practice.

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